(Thesis Synopsis - 5000 words)

A STUDY ON VARIOUS DISCRIMINATION TASKS AND THEIR IMPLICATIONS IN QUANTUM INFORMATION PROCESSING

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Submitted by

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List of Publications

*1. *Local quantum state marking;*

Samrat Sen, Edwin Peter Lobo, Sahil Gopalkrishna Naik, Ram Krishna Patra, Tathagata Gupta, Subhendu B. Ghosh, Sutapa Saha, Mir Alimuddin, Tamal Guha, Some Sankar Bhattacharya, and Manik Banik;

Phys. Rev. A **105**, 032407 (2022). [**SS 1**]

*2. *Timelike correlations and quantum tensor product structure;*

Samrat Sen, Edwin Peter Lobo, Ram Krishna Patra, Sahil Gopalkrishna Naik, Anandamay Das Bhowmik, Mir Alimuddin, and Manik Banik;

Phys. Rev. A **106**, 062406 (2022). [**SS 2**]

*3. *Distilling Nonlocality in Quantum Correlations;*

Sahil Gopalkrishna Naik, Govind Lal Sidhardh, **Samrat Sen**, Arup Roy, Ashutosh Rai, and Manik Banik;

Phys. Rev. Lett. **130**, 220201 (2023). [**SS 3**]

4. *Mutually unbiased balanced functions and generalized random access codes;*

Vaisakh M, Ram krishna Patra, Mukta Janpandit, **Samrat Sen**, Manik Banik, and Anubhav Chaturvedi;

Phys. Rev. A **104**, 012420 (2021). [**SS 4**]

5. *Composition of Multipartite Quantum Systems: Perspective from Timelike Paradigm;*

Sahil Gopalkrishna Naik, Edwin Peter Lobo, **Samrat Sen**, Ram Krishna Patra, Mir Alimuddin, Tamal Guha, Some Sankar Bhattacharya, and Manik Banik;

Phys. Rev. Lett. **128**, 140401 (2022). [**SS 5**]

6. *Certifying beyond quantumness of locally quantum no-signaling theories through a quantum-input Bell test;*

Edwin Peter Lobo, Sahil Gopalkrishna Naik, **Samrat Sen**, Ram Krishna Patra, Manik Banik, and Mir Alimuddin;

Phys. Rev. A **106**, L040201 (2022) [**SS 6**]

7. *Advantage of Hardy's nonlocal correlation in reverse zero-error channel coding;*

Mir Alimuddin, Ananya Chakraborty, Govind Lal Sidhardh, Ram Krishna Patra,

Samrat Sen, Snehasish Roy Chowdhury, Sahil Gopalkrishna Naik, and Manik Banik;
Phys. Rev. A **108**, 052430 (2023). [**SS 7**]

8. *Principle of Information Causality Rationalizes Quantum Composition;*
Ram Krishna Patra, Sahil Gopalkrishna Naik, Edwin Peter Lobo, **Samrat Sen**, Govind Lal Sidhardh, Mir Alimuddin, and Manik Banik;
Phys. Rev. Lett. **130**, 110202 (2023). [**SS 8**]

9. *Classical analogue of quantum superdense coding and communication advantage of a single quantum system;*
Ram Krishna Patra, Sahil Gopalkrishna Naik, Edwin Peter Lobo, **Samrat Sen**, Tamal Guha, Some Sankar Bhattacharya, Mir Alimuddin, and Manik Banik;
Quantum **8**, 1315 (2024). [**SS 9**]

* Thesis related works

List of Communicated Work

1. *Overcoming Traditional No-Go Theorems: Quantum Advantage in Multiple Access Channels;*

   Ananya Chakraborty, Sahil Gopalkrishna Naik, Edwin Peter Lobo, Ram Krishna Patra, **Samrat Sen**, Mir Alimuddin, Amit Mukherjee, Manik Banik;

   arXiv: 2309.17263 [**SS 10**]

2. *Asymptotic Birkhoff-Violation in Operational Theories: Thermodynamic Implications and Information Processing;*

   Ananya Chakraborty, Sahil Gopalkrishna Naik, **Samrat Sen**, Ram Krishna Patra, Pratik Ghosal, Mir Alimuddin, Manik Banik;

   arXiv: 2406.08803 [**SS 11**]

3. *Local-Data-Hiding and Causal Inseparability: Probing Indefinite Causal Structures with Cryptographic Primitives;*

   Sahil Gopalkrishna Naik, **Samrat Sen**, Ram Krishna Patra, Ananya Chakraborty, Mir Alimuddin, Manik Banik, Pratik Ghosal;

   arXiv: 2407.20543*v*1 [**SS 12**]

4. *Scalable & Noise-Robust Communication Advantage of Multipartite Quantum Entanglement;*

   Ananya Chakraborty, Ram Krishna Patra, Kunika Agarwal, **Samrat Sen**, Pratik Ghosal, Sahil Gopalkrishna Naik, Manik Banik;

   arXiv:2409.13223 [**SS 13**]

1 Introduction

The age of information, inaugurated by Claude Shannon's foundational work in 1948 [1], established a mathematical basis for quantifying information, including concepts like entropy, data compression, and channel capacity. Shannon's theorems delineate the boundaries of reliable communication over noisy channels, shaping digital communication, data storage, and error correction.

At the core of reliable communication—and, more broadly, any information-processing protocol—lies the essential discrimination task. Extracting information relies on the ability to effectively distinguish the objects encoding that information. Since communication deals with the sending of classified information among parties, it is a matter of security concern. Hence in the presence of adversaries attempting to intercept communication between parties, security is compromised. To facilitate secure communication between parties, the RSA encryption was developed [2]. However, it was challenged by quantum computing's ability to factorize the composite numbers RSA depends on. This vulnerability led to the emergence of quantum cryptography, with the BB84 protocol [3]—inspired by Stephen Wiesner's pioneering ideas [4]—as the first practical quantum protocol. It capitalizes on the quantum property that non-orthogonal states cannot be reliably distinguished, securing BB84 while showcasing the powerful role of quantum discrimination tasks in revealing fundamental aspects of the underlying theory within an information-theoretic framework.

The goal of this thesis is to harness the power of discrimination and construct various such tasks and see their implications in quantum information processing. In this work, we will propose task of local state marking (LSM), which will be shown to be distinct from the vastly explored task of local state distinguishability (LSD). Through this, we will show the local unmarkability is a stronger form of nonlocality than local indistinguishability. On the other hand, we will establish that beyond quantum composite structure can lead to beyond quantum correlations in timelike scenario, hence welcoming new principles to isolate the quantum correlations from the beyond quantum ones. Thereafter, a protocol in the simplest Bell scenario is given which can distill nonlocality to significantly high degree starting from arbitrarily weak quantum nonlocal correlations. It will be shown that is is effective in detecting post quantum correlations as well.

2 A Brief overview of the topic

2.1 The LOCC paradigm

The state discrimination task is investigated within distant laboratory paradigm which is captured by the framework of Local Operations and Classical Communication (LOCC). In this context, spatially

separated parties operate in distant laboratories, wherein a referee selects a state from a known set of multipartite states and distributes individual subsystems to each party. Under this setup, each party is restricted to performing local measurements and can only communicate via classical channels. The LOCC paradigm effectively models such distant laboratory scenarios, where questions of entanglement manipulation of the shared state and the transformation of one state to another are other important considerations. Contrary to Bell nonlocality [5], where separable and thus product states cannot exhibit nonlocality, a discrimination task revealed that quantum nonlocality can indeed exist without entanglement. This was first demonstrated by Asher Peres and William K. Wootters in 1991 in the context of local quantum state discrimination of product states [6]. This was later expanded by Charles H. Bennett and his collaborators in 1999 [7], wherein they showed a stronger result by using orthogonal product states, whereas Peres and Wooters used non-orthogonal ones. The results in [7] show that certain bipartite orthogonal product quantum states, despite being locally preparable by two parties, require a joint measurement for optimal discrimination which cannot be achieved by spatially separated parties in the LOCC framweork, something which is unthinkable in the classcial regime. This is because in classical theory, states which can be distinguished when complete global access is given, can be locally distinguished as well by classically communicating parties. The difference in discrimination probabilities between these scenarios illustrates the nature of the nonlocality in question.

The LOCC framework has become an active area of research, with applications in cryptographic primitives such as data hiding [8, 9] and secret sharing [10, 11, 12, 13, 14], secured key distribution [15, 16] quantum cryptography [17].

2.2 Quantum theory: a GPT

Quantum theory can be seen as a special instance of a GPT. State space of a d-level quantum system associated with complex Euclidean space $\mathbb{C}^d$ is the set of density operators acting on $\mathbb{C}^d$, *i.e.*, $\Omega(\mathbb{C}^d) \equiv \mathcal{D}(\mathbb{C}^d)$. The set $\mathcal{D}(\mathbb{C}^d)$ is a convex compact set embedded in $\mathbb{R}^{d^2-1}$. The unnormalized state cone is the set of all non-negative operators $\mathcal{P}(\mathbb{C}^d) := \{\lambda\rho \mid \lambda \geq 0 \ \& \ \rho \in \mathcal{D}(\mathbb{C}^d)\}$, which is also the unnormalized effect cone. In other words, quantum theory is self dual.

The minimal composition of two quantum systems associated with Hilbert spaces $\mathbb{C}^{d_A}$ and $\mathbb{C}^{d_B}$ allows only separable state we call it as SEP composition and denote the resulting system as the triplet

$S^{AB}_{\text{SEP}} \equiv \left[\mathbb{C}^{d_A}, \mathbb{C}^{d_B}, \bigotimes_{\text{SEP}}\right]$. Formally the state space for the SEP composition is given by

$$\Omega_{\text{SEP}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}) := \Bigg\{ \rho_{AB} = \sum_i p_i \rho^i_A \otimes \rho^i_B \mid p_i \geq 0 \\ \& \ \sum_i p_i = 1;\ \rho^i_A \in \mathcal{D}(\mathbb{C}^{d_A}),\ \rho^i_B \in \mathcal{D}(\mathbb{C}^{d_B}) \Bigg\}.$$

Since $\Omega_{\text{SEP}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B})$ contains only separable states, the corresponding effect space $\mathcal{E}_{\text{SEP}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B})$ contains effects that are not allowed in quantum theory. Entanglement witness operators yielding positive probability on separable states are valid effects in this composition although they are not allowed in quantum theory. On the other extreme, the maximal composition, which we will call $\overline{\text{SEP}}$, and the resulting system denote as $S^{AB}_{\overline{\text{SEP}}} \equiv \left[\mathbb{C}^{d_A}, \mathbb{C}^{d_B}, \bigotimes_{\overline{\text{SEP}}}\right]$, has the state space

$$\Omega_{\overline{\text{SEP}}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}) := \Big\{ W_{AB} \in \text{Herm}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}) \mid \\ \text{tr}(W_{AB}) = 1, \text{tr}[W_{AB}(\pi_A \otimes \pi_B)] \geq 0 \\ \forall\ \pi_A \in \mathcal{P}(\mathbb{C}^{d_A}), \pi_B \in \mathcal{P}(\mathbb{C}^{d_B}) \Big\}.$$

Here $\text{Herm}(\mathcal{X})$ denotes the set of Hermitian operators acting on the space $\mathcal{X}$, and normalization demands $\text{tr}(W_{AB}) = 1\ \forall\ W_{AB} \in \Omega_{\overline{\text{SEP}}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B})$. The unnormalized effect cone corresponding to $\mathcal{E}_{\overline{\text{SEP}}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B})$ is identical to the unnormalized state cone corresponding to the set $\Omega_{\text{SEP}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B})$. For the quantum case $S^{AB}_{Q} \equiv \left[\mathbb{C}^{d_A}, \mathbb{C}^{d_B}, \bigotimes_{Q}\right]$ we have $\Omega_Q(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}) = \mathcal{D}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$, and the effect cone is identical to the state cone which represents the self duality of quantum theory. The following set inclusion relations are immediate

$$\Omega_{\text{SEP}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}) \subset \Omega_Q(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}) \subset \Omega_{\overline{\text{SEP}}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}),$$
$$\mathcal{E}_{\overline{\text{SEP}}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}) \subset \mathcal{E}_Q(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}) \subset \mathcal{E}_{\text{SEP}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}).$$

In between SEP and $\overline{\text{SEP}}$, many other compositions can be defined by appending/deducting suitable states/effects. Among these, quantum composition is the only one that is self dual.

2.3 Bell's theorem:

For any theory that admits an underlying local hidden variable description, in 1964 [18] Bell derived an inequality for bipartite. A more precise formulation of this inequality is presented by J. Clauser *et. al.* in the Ref. [19]. Consider two spatially separated parties Alice and Bob sharing any arbitrary correlation. Let $\{A_0, A_1\}$ be the measurements that Alice performs randomly on her subsystem with

a denoting the outcome which can be either $+1$ or -1. Likewise for Bob. The joint probability that Alice and Bob obtain outcome a and b if Alice measures A_i on her system and Bob measures B_j on his system, respectively is denoted by $p(ab|A_i, b_j)$. Existence of a hidden variable description implies $p(ab|A_i, B_j) = \int \mu(\lambda)p(ab|A_i, B_j, \lambda)d\lambda$, where, $\mu(\lambda)$ is the probability distribution over the hidden variable λ. If the underlying hidden variable model satisfies both reality and locality, the observed correlation at the operational level must satisfy the following inequality (referred to as CHSH inequality)[20]

$$|\langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle| \leq 2,$$

where, $\langle A_iB_j\rangle = \sum_{a,b=-1,+1} abp(ab|A_iB_j)$ denotes the expectation value corresponding to the A_i and B_j measurements which can be calculated from the input-output statistics $p(ab|A_i, b_j)$. Although quantum mechanics violates this bound, Tsirelson [21] showed that the maximal Bell-CHSH value that can be obtained from quantum mechanics is $2\sqrt{2}$. The implication is profound. Local Realism - the world-view deeply rooted in our classical way of thinking cannot be simultaneously true in any hidden variable model that reproduces quantum statistics. .

2.4 Nonlocality without Inequality:

The fact established by Bell's theorem was also shown later by Lucien Hardy from a different line of argument [22, 23]. Hardy's argument is as follows: like the Bell scenario, consider two spatially separated parties, say Alice and Bob, who are sharing some bipartite composite system. Each of them perform measurements on their respective subsystems. Alice's chosen measurements are either A_0 or A_1, Bob's chosen measurements are either B_0 or B_1. The outcome of every measurement is either $+1$ or -1. From the set of 16 joint probabilities in any such scenario, for Hardy's argument four joint probabilities are chosen:

$$\begin{aligned} p(+1,+1|A_0B_0) &= q\ (>0) \\ p(+1,+1|A_0B_1) &= 0 \\ p(+1,+1|A_1B_0) &= 0 \\ p(-1,-1|A_1B_1) &= 0 \end{aligned} \tag{1}$$

Now let us assume that the first condition holds, then for some local deterministic hidden variable λ (such λ must exist since $q > 0$), $A_0^\lambda = +1$ and $B_0^\lambda = +1$. From the second and third condition, it implies $B_1^\lambda = -1$ and $A_1^\lambda = -1$ which is in contradiction with the fourth condition. Every non-

maximally entangled pure two-qubit state shows Hardy's nonlocality, while not the maximally one [24, 25]. Unlike Bell-CHSH violation, the mixed states of two-qubit systems cannot provide Hardy's nonlocality [26].

3 A brief summary of my works

3.1 Local quantum state marking [SS 1]

While mutually orthogonal quantum states can always be perfectly distinguished, challenges arise in multipartite systems, where parties are restricted to local quantum operations assisted by classical communication (LOCC). This defines the problem of local state discrimination (LSD) [27, 28, 29, 30, 31, 32, 33, 34], which has been extensively studied over the past two decades, leading to many significant results [35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50], with applications in various tasks [51, 8, 52, 10, 53]. In addition to LSD and general quantum state discrimination [54, 55, 56], other discrimination tasks such as channel and process discrimination have been explored [57, 58, 59, 60], inspiring new information protocols [61, 62, 63, 64, 65].

Here, we introduce a new variant of discrimination, called local state marking (LSM). In LSM, a randomly chosen subset of multipartite states is given to spatially separated parties, who must identify the states using LOCC. For a given set of multipartite states $\mathcal{S}$, we define a class of tasks called m-LSM, where $1 \leq m \leq |\mathcal{S}|$. The task of 1-LSM corresponds to LSD, while $|\mathcal{S}|$-LSM is simply LSM.

We show that LSM is distinct from LSD: local distinguishability implies local markability, but the reverse is not always true. We provide an example of mutually orthogonal states that are locally markable but not distinguishable, as well as examples of orthogonal states that can neither be distinguished nor marked perfectly under LOCC. We also examine the relationships between different m-LSM tasks. Additionally, we explore entanglement-assisted local marking, where additional entanglement enables the marking of otherwise unmarkable states. Notably, in this case, the entanglement can be returned (partially or fully) after the marking task is completed.

Definition 1. *[m-LSM] m number of states chosen randomly from a known set of pairwise orthogonal N-party quantum states $\mathcal{S} \equiv \{|\psi_j\rangle \mid \langle\psi_i|\psi_j\rangle = \delta_{ij}\} \subset \bigotimes_{i=1}^{N} \mathbb{C}_{A_i}^{d_i}$ are distributed among spatially separated parties without revealing the identity of each state. The m-LSM task is to perfectly identify/mark each of the states under the operational paradigm of LOCC.*

In Definition 1, m can take values from 1 to $|\mathcal{S}|$ and accordingly they constitute different discrimination tasks

3.1.1 Results

We will start the technical part of this article by establishing some generic results.

Lemma 1. *For a set of multipartite states $\mathcal{S}$, perfect $(|\mathcal{S}|-2)$-LSM always implies perfect LSM.*

Proof. Perfect $(|\mathcal{S}|-2)$-LSM of the set $\mathcal{S}$ implies that given arbitrary $(|\mathcal{S}|-2)$ states from the set, they can be marked locally. So we are left with two more states to identify locally. According to a standard result by Walgate *et al.*, any two multipartite pure orthogonal states can be distinguished locally [28], which proves our claim. □

While proof of Lemma 1 follows straightforwardly from the result of Walgate *et al.*, in the next we establish a rather nontrivial thesis.

Theorem 1. *For a set of multipartite states $\mathcal{S}$, perfect LSD (i.e. 1-LSM) always implies perfect LSM (i.e. $|\mathcal{S}|$-LSM).*

Corollary 1. *For a set of multipartite states $\mathcal{S}$, perfect m-LSM always implies perfect m'-LSM, where $1 \leq m \leq m'(:= nm) \leq |\mathcal{S}|$ with $n \in \mathbb{N}$.*

Corollary 2. *For a set of multipartite states $\mathcal{S}$ containing only product states, perfect m-LSM always implies perfect m'-LSM, where $1 \leq m \leq m' \leq |\mathcal{S}|$.*

We note in passing that m-LSM does not trivially imply $(m-1)$-LSM for product states. In the m-LSM task, m states are accessible to the parties in order to mark their identities. However, for $(m-1)$-LSM, the number of accessible states reduce to $(m-1)$ and no trivial inferences can be drawn. Although we were unable to prove for product states, whether m-LSM implies (m-1)-LSM or not, if this really were the case, then the consequences would be very exciting. Using the contrapositive of the statement that m-LSM implies (m-1)-LSM, we would have thus given a protocol for constructing locally indistinguishable states in higher dimensions starting from states like Bennett's UPBs for which perfect LSD is not possible.

Our next result, however, establishes that the converse statement of Theorem 6 does not hold in general and we show this by providing an explicit example wherein given a set of entangled states, known to be locally indistinguishable, the task of LSM is still possible, and that too with a substantial amount of surplus entanglement shared between the distant parties at the end of the protocol.

Theorem 2. *Perfect LSM of a given set of states $\mathcal{S}$ does not necessarily imply perfect LSD of $\mathcal{S}$.*

The proof is constructive. We provide a set of pairwise orthogonal states that can be perfectly marked under LOCC but does not allow perfect local distinguishability. To this aim consider the set of states $\mathcal{X}_4 \equiv \{|\chi_i\rangle\}_{i=1}^4 \subset \mathbb{C}_A^4 \otimes \mathbb{C}_B^4$ shared between Alice and Bob, where

$$
\begin{aligned}
&|\chi_1\rangle := |\phi^+\rangle_{A_1B_1} \otimes |\phi^+\rangle_{A_2B_2}, \ |\chi_2\rangle := |\phi^-\rangle_{A_1B_1} \otimes |\phi^-\rangle_{A_2B_2}, \\
&|\chi_3\rangle := |\psi^+\rangle_{A_1B_1} \otimes |\phi^-\rangle_{A_2B_2}, \ |\chi_4\rangle := |\psi^-\rangle_{A_1B_1} \otimes |\phi^-\rangle_{A_2B_2}, \\
&\text{with} \quad |\phi^\pm\rangle := \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}, \quad |\psi^\pm\rangle := \frac{|01\rangle \pm |10\rangle}{\sqrt{2}},
\end{aligned}
$$

and A_1, A_2 subsystems are with Alice while B_1, B_2 are with Bob.

Proposition 1. *The two qubit Bell basis* $\mathcal{B}_4 \equiv \{|b_1\rangle := |\phi^+\rangle, |b_2\rangle := |\phi^-\rangle, |b_3\rangle := |\psi^+\rangle, |b_4\rangle := |\psi^-\rangle\} \subset \mathbb{C}^2 \otimes \mathbb{C}^2$ *is locally unmarkable.*

Proposition 2. *Consider any set of maximally entangled states* $\mathcal{B}_K(d) := \{|b_i\rangle \mid \langle b_i|b_j\rangle = \delta_{ij}\}_{i=1}^{K} \subset \mathbb{C}^d \otimes \mathbb{C}^d$. *The set is locally unmarkable whenever* $K! > d^K$.

Proposition 3. *There exists a* $(2, 1)$ *entanglement catalytic perfect protocol for LSM of the set* $\mathcal{B}_4$.

Proposition 4. *Any three Bell states of the two-qubit system is unmarkable under one-way LOCC.*

Corollary 3. *Perfect 2-LSM of the set* $\mathcal{B}_3$ *is not possible even under two-way LOCC protocol.*

Proposition 5. *There exists a* $(1, 1)$ *entanglement catalytic protocol for perfect LSM of the set* $\mathcal{B}_3$.

Except Corollary 2, the present work does not provide much insight for the local state marking of ensembles containing only product states. Does there exist such a product ensemble that cannot be marked locally? If yes, would it imply a stronger notion of *nonlocality without entanglement* [27]? In recent past, this phenomena of *nonlocality without entanglement* has been studied in the generalized probabilistic theory framework [48]. It might be interesting to extend the study of LSM in this framework.

3.2 Timelike correlations and quantum tensor product structure [SS 2]

The tensor product postulate in quantum mechanics, often referred to as the "zeroth" axiom [66], states that the Hilbert space of a composite system is the tensor product of its components' Hilbert spaces [67, 68, 69]. A recent study, however, derives this postulate from the state and measurement postulates instead of treating it as independent [70]. Within this structure, Gleason's theorem assigns state spaces to composite systems, including quantum states as a subset [71, 72, 73]. Various models can describe composite state and effect spaces that align with quantum theory, providing insights into quantum information processing.

Generalized probabilistic theory (GPT) [74, 75, 76, 77, 78, 79] is a useful framework for studying these models. Physical constraints like no-signaling and local tomography limit state spaces between two extremes [80, 81, 82, 83]: the minimal tensor product containing only separable states and the maximal tensor product including beyond-quantum states compatible with Gleason's theorem. Effect spaces follow the "no-restriction" hypothesis, which includes all mathematically consistent effects [84].

A key question is whether these models can lead to stronger-than-quantum correlations. For bipartite systems, Barnum *et al.* [85] show that no such composition yields beyond-quantum correlations in

Bell-type scenarios. However, stronger-than-quantum correlations are possible in multipartite systems [86] and even in bipartite cases when using quantum input-classical output scenarios [87, 88]. While beyond-quantum states are excluded, the scenario requires trustworthy verifiers for predetermined quantum inputs [88]. In a different approach, Naik *et al.* [89] show that the minimal composition can yield stronger-than-quantum correlations in a timelike scenario, such as a communication game between two time-separated players. Although the minimal composition cannot produce nonlocal correlations, its beyond-quantum effects result in stronger-than-quantum timelike correlations.

Contrary to intuition, we show that beyond-quantum effects are not necessary for beyond-quantum timelike correlations. Specifically, the maximal composition, which includes beyond-quantum states but only product effects, also generates these correlations. While Barnum *et al.* [85] rule out spacelike correlations in the bipartite maximal composition, we demonstrate that timelike correlations reveal its beyond-quantum nature. However, we also prove a no-go result: despite allowing stronger timelike correlations, the classical information capacity of maximal composition models does not exceed that of quantum systems. This leads to a general result on the information capacity of composite systems in the GPT framework.

Pairwise distinguishability game $\mathcal{P}_D^{[n]}$:– The game involves two players (Alice and Bob) and a Referee. In each run of the game, the Referee provides a classical message η to Alice, randomly chosen from some set of messages $\mathcal{N}$, where $|\mathcal{N}| := n$. In the same run Bob is asked a question $\mathbb{Q}(\eta, \eta')$ – whether the message given to Alice is η or η', where $\eta' \neq \eta$. The winning condition demands Bob answer all questions correctly. Alice can help Bob by sending some information about the message she received. It is not hard to see that perfect winning demands Alice to encode the message on the states of some physical system that are pairwise distinguishable. With this game we are now in a position to prove one of our main results.

Theorem 3. *The game* $\mathcal{P}_D^{[8]}$ *cannot be won if Alice uses the system* $[\mathbb{C}^2, \mathbb{C}^2, \bigotimes_Q]$ *to encode her message whereas* $[\mathbb{C}^2, \mathbb{C}^2, \bigotimes_{\overline{SEP}}]$ *system yields a perfect winning strategy.*

Corollary 4. *The information dimension of the system* $[\mathbb{C}^2, \mathbb{C}^2, \bigotimes_{\overline{SEP}}]$ *is at least* 8*, i.e.* $\mathbb{I}[\mathbb{C}^2, \mathbb{C}^2, \bigotimes_{\overline{SEP}}] \geq 8$.

Proposition 6. *Every POPT state* $W_{AB} \in \Omega_{\overline{SEP}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B})$ *can be written as* $(\mathcal{I}_A \otimes \Lambda_{R\to B})(\rho_{AR})$ *where* Λ *is a positive, unital map and* $\rho_{AR} \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_R}$ *is a pure quantum state independent of* W_{AB}.

Theorem 4. *The Operational Dimension of the system* $[\mathbb{C}^{d_A}, \mathbb{C}^{d_B}, \bigotimes_{\overline{SEP}}]$ *is* $d_A d_B$.

Proposition 7. *The operational dimension of any bipartite composition (with normalized state space denoted as* Ω_{AB}*) of two elementary quantum systems* $\mathbb{C}^{d_A}$ *and* $\mathbb{C}^{d_B}$ *is* $d_A d_B$ *if* $(\mathbf{1}_{AB} - W_{AB})$ *lies within*

the unnormalised state cone $\forall\ W_{AB} \in \Omega_{AB}$.

Proposition 8. *Let* $\mathcal{S}_{AB} \equiv (\Omega_{AB}, \mathcal{E}_{AB})$ *be a composite systems consisting two elementary systems* $\mathcal{S}_A \equiv (\Omega_A, \mathcal{E}_A)$ *and* $\mathcal{S}_B \equiv (\Omega_B, \mathcal{E}_B)$ *with operational dimensions* N_A *and* N_B *respectively. The operational dimension of* $\mathcal{S}_{AB}$ *is* $N_A N_B$ *if* $\exists\ \omega'_{AB} \in \Omega_{AB}$ *such that* $(N_A N_B \omega'_{AB} - \omega_{AB})$ *is an unnormalized state* $\forall\ \omega_{AB} \in \Omega_{AB}$.

Form Theorem 6 and Theorem 7 we can conclude that $\Delta[\mathbb{C}^2, \mathbb{C}^2, \bigotimes_{\overline{\text{SEP}}}] \geq 4$, where Δ refers to the dimension mismatch of the theory. On the other hand, we can also conclude that the gap between the information dimensions of the systems $[\mathbb{C}^2, \mathbb{C}^2, \bigotimes_{\overline{\text{SEP}}}]$ and $[\mathbb{C}^2, \mathbb{C}^2, \bigotimes_Q]$ is at least 4, *i.e.*,

$$\mathbb{I}\left[\mathbb{C}^2, \mathbb{C}^2, \otimes_{\overline{\text{SEP}}}\right] - \mathbb{I}\left[\mathbb{C}^2, \mathbb{C}^2, \otimes_Q\right] \geq 4.$$

Our next result shows that this gap can be increased further by considering more number of elementary systems.

Theorem 5. *The Information Dimension of the system* $[\mathbb{C}^2, \mathbb{C}^2, \mathbb{C}^2, \bigotimes_{\overline{SEP}}]$ *is at least* 24.

Theorem 8 thus establishes that the $\mathcal{P}_D^{[24]}$ game can be won with three elementary qubits if the $\overline{\text{SEP}}$ composition is considered among them, whereas if we consider quantum composition five elementary qubits are required.

A natural question then is to ask what other compositions can be ruled out using dimension mismatch. Another interesting direction to explore is by relaxing the assumption of quantum subsystems. Propositions 7 and 8 provide some preliminary results in the GPT framework which may be useful in this regard. Our study forms an important piece of the quantum reconstruction program in which we seek to derive quantum theory from physical principles [74, 75, 76].

3.3 Distilling nonlocality in quantum correlations [SS 3]

The study of nonlocality distillation has two key implications: (i) practical, where distilling nonlocal correlations can improve the efficiency and security of information networks, and (ii) foundational, aiming to identify post-quantum correlations to understand quantum theory's uniqueness within generalized probabilistic theories. In Ref.[90], Forster *et al.* proposed a protocol for distilling stronger nonlocality from weakly nonlocal systems, inspiring numerous works on the topic [91, 92, 93, 94, 95, 96, 97]. Most research on nonlocality distillation has focused on post-quantum correlations [91, 92, 95, 98]. Only a few protocols, such as [90, 98], have successfully distilled some quantum correlations, but their nonlocal strength remains low.

In this letter, we propose a new framework for nonlocality distillation that addresses the limitations of existing protocols. Unlike earlier work, we focus on finding efficient protocols for distilling quantum correlations. Instead of starting with general NS correlations, we take a bottom-up approach by beginning with weak quantum nonlocal correlations and developing a distillation protocol. Since the set of quantum correlations is closed under *wirings* [99, 100], the resulting correlations remain quantum. We introduce a simple protocol, the OR-AND protocol, which successfully distills nonlocality in a wide class of weak quantum correlations. Specifically, we use a variant of Hardy's nonlocality test [101] and show that two copies of a quantum Hardy correlation can be distilled through a simple wiring operation, called the OR-AND protocol. This method generalizes to multiple copies and significantly enhances Hardy's success probability. Furthermore, the OR-AND protocol amplifies small violations of the Clauser-Horne-Shimony-Holt (CHSH) inequality [102] to a much higher degree.

Additionally, we demonstrate that correlations arbitrarily close to the extreme points of the local correlation set can always be distilled, proving that distillable quantum correlations occupy a non-zero measure in the full eight-dimensional correlation space. We also show that the OR-AND protocol works efficiently for post-quantum correlations, identifying cases where known information principles, such as nontrivial communication complexity [103] and information causality [104], fail. Lastly, we compare our approach with existing methods, highlighting its novelty and effectiveness in nonlocality distillation.

Distillation of quantum nonlocality– We start by presenting our results on nonlocality distillation of Hardy's correlations, by applying the OR-AND protocol.

Theorem 6. *The OR-AND wiring preserves the structure of quantum Hardy correlations and can efficiently distil the strength of success probability in Hardy's test of nonlocality.*

It turns out that, then any quantum Hardy correlation H_Q, can be represented as

$$\mathrm{H}_Q = c_0 \mathrm{P}_{NL} + \sum_{i=1}^{5} c_i \mathrm{P}_{L_i}; \quad c_i > 0 \ \forall \ i, \quad \text{and} \ \sum_{i=0}^{5} c_i = 1, \tag{2}$$

Proposition 9. *A no-signaling Hardy correlation of the form of Eq.(2) yields Hardy success* $p^{(n)}_{Hardy} = (\frac{c_0}{2} + c_1)^n - c_1^n$, *when its n-copy is wired under OR-AND protocol. The optimal value of distilled Hardy success is given by* $p^{opt}_{Hardy} = \max\{p^{(\mathrm{N})}_{Hardy}, \ p^{(\mathrm{N}+1)}_{Hardy}\}$, *where*

$$\mathrm{N} := \left\lfloor \log\left(\frac{\log c_1}{\log \frac{c_0}{2} + c_1}\right) / \log\left(\frac{\frac{c_0}{2} + c_1}{c_1}\right) \right\rfloor. \tag{3}$$

Theorem 7. *Starting with a quantum correlation with arbitrarily small CHSH nonlocality OR-AND*

wiring can yield Tsirelson gain up to ($\approx$) 39.75%.

Theorem 8. *CHSH nonlocality of any no-singling correlation of the form* $\tilde{\mathrm{C}}(\lambda) = \lambda \mathrm{C} + (1-\lambda) P_{L_1}$, *where* $0 < \lambda \leq 1$ *and* $\mathrm{C} \in \mathrm{ConvexHull}\ \{\mathrm{P}_{NL}, \mathrm{P}_{L_i} \mid i \in \{1, \cdots, 8\}\}$ *can be distilled through OR-AND wiring by choosing the values of* λ *sufficiently small. Furthermore, 2-copy OR-AND distillation is successful for all the* $\tilde{C}(\lambda)$ *correlation boxes whenever* $\lambda < \frac{2}{3} c_0$; *where* c_0 *is the* P_{NL} *fraction in* C.

Distillation of post-quantum nonlocality.– We now proceed to show that OR-AND wiring also has an important proviso in ruling out (unphysical) post-quantum correlations. Checking membership to the NPA hierarchy can become computationally expensive, particularly at higher orders of the hierarchy, while the distillation criteria is far more tractable. For instance, let us consider the correlation:

$$\mathrm{H}'_{NS} = \begin{array}{c} xy/ab \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} 0.0773 & 0.0256 & 0.5599 & 0.3372 \\ 0 & 0.1029 & 0.7804 & 0.1167 \\ 0 & 0.3374 & 0.6372 & 0.0254 \\ 0.1178 & 0.2196 & 0.6626 & 0 \end{pmatrix} \end{array} . \tag{4}$$

One may easily verify that correlation given in Eq.(4) is post-quantum by observing that after 2-copy distillation using OR-AND protocol, the Hardy success goes up to 0.0925 (a value beyond the maximum possible success probaility in quantum mechanics). On the other hand, for the considered correlation, tests like known necessary conditions for violating non-trivial computational complexity and information causality principle fails to detect its post-quantumness. However, unlike the first example, on considering NPA criteria, a membership test into the second tier of the NPA hierarchy is required to establish the post-quantumness of this correlation. Along similar lines, one may imagine post-quantum correlations like Eq.(4) that lies at further deeper levels of the NPA-hierarchy, while its post-quantumness may be conveniently detected via efficient nonlocality distillation protocols.

As for future, it would be interesting to explore the full potential of our generic framework proposed here in distilling quantum nonlocal correlations. In particular, obtaining some bound on the relative volume of the quantum correlations in the correlation space that can be distilled under OR-AND wiring would be interesting. Furthermore a generalization of this protocol for higher input-output as well as in multiparty scenario might be of great use.

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